SPACES FOR WHICH THE STONE-WEIERSTRASS THEOREM HOLDS[1]

BY

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If X is a topological space, a subset A of $C(X)$, the set of bounded continuous real functions on X, is said to separate the points of X if for every pair x, y of distinct points of X there is a function f in A with $f(x) \neq f(y)$. A space X is called completely Hausdorff if $C(X)$ separates the points of X. The Stone-Weierstrass theorem states: If X is a compact Hausdorff space, and if A is a subalgebra of $C(X)$ which (i) separates the points of X and (ii) contains the constants, then A is uniformly dense in $C(X)$[2]. In the following, we shall say the Stone-Weierstrass theorem holds for a space X provided that X is completely Hausdorff, and that every subalgebra of $C(X)$ which satisfies (i) and (ii) is uniformly dense in $C(X)$.

An extension space of a topological space X is a pair (Y, h), where Y is a topological space, h is a homeomorphism of X into Y, and $h(X)$ is dense in Y; if h is the identity map, the reference to h is omitted, and Y itself is called an extension space of X; (Y, h) is called proper if $h(X)$ is a proper subset of Y. We shall call a completely Hausdorff space X completely Hausdorff-complete if and only if X has no proper extension space (Y, h) such that Y is a completely Hausdorff space.

A filter on a space X is called completely regular provided that it has a base $\mathscr{B}$ of open sets such that for each B in $\mathscr{B}$, there is a set $B' \subset B$ in $\mathscr{B}$ and a function $f \in C(X)$ which maps X into $[0, 1]$, is 0 on B', and is 1 on $X - B$. In [3] Banaschewski proved that the Stone-Weierstrass theorem holds for a completely Hausdorff space X if and only if every completely regular filter $\mathscr{F}$ on X has the property that $\bigcap \{F \mid F \in \mathscr{F}\} \neq \varnothing$. Using this result, we shall prove the following:

THEOREM 1. *The Stone-Weierstrass theorem holds for a completely Hausdorff space X if and only if X is completely Hausdorff-complete.*

Proof. Suppose that X has a proper extension space (Y, h) such that Y is a completely Hausdorff space. Fix a point y in $Y - h(X)$, and let $Z = \{f \in C(h(X)) \mid f = g \mid h(X)$, where g is in $C(Y)$, and $g(y) = 0\}$. For each $f \in Z$ and number $0 < t$, define $W(f, t) = \{z \mid -t < f(z) < t\}$, and let $\mathscr{F}$ be the filter on $h(X)$ generated by the collection of all finite intersections of elements of $\{W(f, t) \mid f \in Z, 0 < t\}$. It is not difficult to see that $\mathscr{F}$ is a completely regular filter on $h(X)$: take $\bigcap \{W(f_i, t_i) \mid i = 1, \ldots, n\} \in \mathscr{F}$; for each integer i, $1 \leqq i \leqq n$, choose a number

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[2] For definitions not given here, see [8] or [13].

$0 < s_i < t_i$; since the real line is normal, for each i there exists a function $g_i \in C(h(X))$ such that $g_i(h(X)) \subset [0, 1]$, $g_i(W(f_i, s_i)) = 0$, and $g_i(h(X) - W(f_i, t_i)) = 1$; then $g \equiv \max\{g_i\} \in C(h(X))$, $\bigcap\{W(f_i, s_i)\} \subset \bigcap\{W(f_i, t_i)\}$, $g(\bigcap\{W(f_i, s_i)\}) = 0$, and $g(h(X) - \bigcap\{W(f_i, t_i)\}) = 1$. Thus $h^{-1}(\mathscr{F})$ is a completely regular filter on X, but since Y is completely Hausdorff, $\varnothing = \bigcap\{F \mid F \in \mathscr{F}\}$, so $\varnothing = \bigcap\{h^{-1}(F) \mid F \in \mathscr{F}\}$, which contradicts the assumption that the Stone-Weierstrass theorem holds for X.

Conversely, suppose that the Stone-Weierstrass theorem does not hold for X. Then there exists (Zorn's lemma) a maximal completely regular filter $\mathscr{G}$ on X such that $\bigcap\{G \mid G \in \mathscr{G}\} = \varnothing$. Let Y be the set $X \cup \{\mathscr{G}\}$, topologized as follows: a set $O \subset Y$ is open if and only if (i) $O \cap X$ is open in X, and (ii) $\mathscr{G} \in O$ implies $0 \cap X \in \mathscr{G}$. Clearly Y is a proper extension space of X. We show that Y is completely Hausdorff. Let $x \in X$. As $\bigcap\{G \mid G \in \mathscr{G}\} = \varnothing$, there is a function $f \in C(X)$ such that $f\colon X \to [0, 1]$, $f(x) = 1$, and $f(G) = 0$, some $G \in \mathscr{G}$. Define $f'\colon Y \to [0, 1]$ by $f' = f$ on X, and $f'(\mathscr{G}) = 0$. Then $f' \in C(Y)$, and $f'(x) \neq f'(\mathscr{G})$. That $C(Y)$ separates the points of X is a consequence of the fact that each function in $C(X)$ has an extension in $C(Y)$.

Let $g \in C(X)$. Since $[g(X)]^-$ is compact, there is a number t in the adherence of the filter base $g(\mathscr{G})$. Since $\mathscr{G}$ is a maximal completely regular filter, the inverse image under g of each neighborhood of t is an element of $\mathscr{G}$, i.e., $g(\mathscr{G})$ converges to t. Thus the function g' defined by $g' = g$ on X, and $g'(\mathscr{G}) = \lim g(\mathscr{G})$ is an extension of g in $C(Y)$.

It would be interesting to know when the Stone-Weierstrass theorem holds for the product of a collection of topological spaces. As far as the author knows, the following problem is unsolved: If $\{X_a \mid a \in A\}$ is a collection of topological spaces such that the Stone-Weierstrass theorem holds for each X_a, $a \in A$, does the Stone-Weierstrass theorem hold for $\prod\{X_a \mid a \in A\}$? The next theorem gives a partial answer to this question.

THEOREM 2. *If X_1 is a compact Hausdorff space, and if X_2 is a space for which the Stone-Weierstrass theorem holds, then the Stone-Weierstrass theorem holds for $X_1 \times X_2$. If $\{X_a \mid a \in A\}$, is a collection of spaces with the property that the Stone-Weierstrass theorem holds for their product $X = \prod\{X_a \mid a \in A\}$, then the Stone-Weierstrass theorem holds for each X_a, $a \in A$.*

Proof. For the first part, assume X_1 and X_2 have the given properties. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be distinct points of $X_1 \times X_2$. Then $x_i \neq y_i$, $i = 1$, or $i = 2$, so there is a function f in $C(X_i)$ such that $f(x_i) \neq f(y_i)$. Hence $f \circ \mathrm{pr}_i$ is in $C(X_1 \times X_2)$ and satisfies $f \circ \mathrm{pr}_i(x) \neq f \circ \mathrm{pr}_i(y)$. Suppose that $\mathscr{C}$ is a completely regular filter on $X_1 \times X_2$. We shall show that $\bigcap\{C \mid C \in \mathscr{C}\} \neq \varnothing$. The filter generated by $\mathrm{pr}_2(\mathscr{C})$ has a base consisting of open sets since pr_2 is an open mapping. Furthermore, it is completely regular, for take an open set $C \in \mathrm{pr}_2(\mathscr{C})$, say $C = \mathrm{pr}_2(B)$, where $B \in \mathscr{C}$. There exists a set $B' \subset B$ in $\mathscr{C}$ and a function g in $C(X_1 \times X_2)$ which maps $X_1 \times X_2$ into $[0, 1]$, is 0 on B', and is 1 on $(X_1 \times X_2) - B$. For each number $t \in (0, 1)$, let U_t be the open set $\mathrm{pr}_2(g^{-1}([0, t)))$. Then whenever $0<s<t<1$, we have $\mathrm{pr}_2(B')\subset \bar{U}_s \subset U_t\subset \bar{U}_t\subset C$, for since X_1 is compact, pr_2 is a closed mapping. Define $h: X_2 \to [0, 1]$ by $h(x)=0$ if $x \in U_t$ for all $t \in (0, 1)$, and $h(x)=\sup\{t \mid x \notin U_t\}$ otherwise. If $0<a<1$, $h^{-1}([0, a))=\bigcup \{U_t \mid t<a\}$ and $h^{-1}((a, 1])=\bigcup \{X_2-\bar{U}_t \mid t>a\}$ are open sets. Hence $h \in C(X_2)$, $h(\mathrm{pr}_2(B'))=0$, and $h(X_2-C)=1$. By hypothesis the Stone-Weierstrass theorem holds for X_2; thus there is a point $x \in \bigcap \{\mathrm{pr}_2(C) \mid C \in \mathscr{C}\}$. Then $\mathscr{D}=\{C \cap (X_1 \times \{x\}) \mid C \in \mathscr{C}\}$ is a filter base on $X_1 \times X_2$. $\mathrm{pr}_1(\mathscr{D})$, a filter base on a compact space, has an adherent point y. Therefore, for $C \in \mathscr{C}$ and an arbitrary open set $Y\subset X_1$, $y \in Y$ implies $C \cap (Y\times\{x\})=C \cap (X_1 \times \{x\}) \cap (Y\times X_2)\neq \varnothing$, i.e., (y, x) is an adherent point of $\mathscr{C}$. Since the adherence of a completely regular filter is the same as the intersection of all the sets belonging to it, $(y, x) \in \bigcap \{C \mid C \in \mathscr{C}\}$.

For the second statement, we assume the Stone-Weierstrass theorem holds for $X = \prod \{X_a \mid a \in A\}$ and consider a factor space X_b.

Let x_b and y_b be distinct points of X_b. For each $a \in A-\{b\}$, fix $z_a \in X_a$, and define $x_a=y_a=z_a$. Set $x=(x_a \mid a \in A)$ and $y=(y_a \mid a \in A)$. $x\neq y$, so there is a function $f\in C(X)$ such that $f(x)\neq f(y)$. Define $Z_b=X_b$, $Z_a=\{z_a\}$ if $a\neq b$, and $Z=\prod \{Z_a \mid a \in A\}$. Let $i=\mathrm{pr}_b|Z^{-1}$ and $f'=f \circ i$. Then $f' \in C(X_b)$, and $f'(x_b)\neq f(y_b)$.

Let $\mathscr{F}$ be a completely regular filter on X_b, and consider the filter generated by $\mathrm{pr}_b^{-1}(\mathscr{F})$. Take $G=\mathrm{pr}_b^{-1}(F)$, F a set in $\mathscr{F}$. There is a function $g \in C(X_b)$ and an open set $F'\subset F$ in $\mathscr{F}$ such that $g(F')=0$, $g(X_b-F)=1$, and $g(X_b)\subset [0, 1]$. The open set $G'=\mathrm{pr}_b^{-1}(F')\subset G$ is in $\mathrm{pr}_b^{-1}(\mathscr{F})$, and the function $g \circ \mathrm{pr}_b \in C(X)$ satisfies $g \circ \mathrm{pr}_b(G')=0$, $g \circ \mathrm{pr}_b(X-G)=1$, and $g \circ \mathrm{pr}_b(X)\subset [0, 1]$. Hence the filter generated by $\mathrm{pr}_b^{-1}(\mathscr{F})$ is completely regular and has an adherent point x. $\mathrm{pr}_b(x) \in \bigcap \{F \mid F \in \mathscr{F}\}$.

By similar reasoning one can prove:

THEOREM 3. *Let X be a space for which the Stone-Weierstrass theorem holds, and suppose that R is an equivalence relation on X. Then the Stone-Weierstrass theorem holds for the quotient space X/R if and only if X/R is completely Hausdorff.*

Although the Stone-Weierstrass theorem does not hold for every completely Hausdorff space, it can be shown that every completely Hausdorff space has an extension space for which it does hold.

Let X be a completely Hausdorff space, and let $\mathscr{M}$ be the set of all maximal completely regular filters on X which have empty adherences. We shall denote by X' the topological space whose points are the elements of $X \cup \mathscr{M}$ and whose open sets are generated by $\{V^* \mid V$ is open in $X\}$, where $V^*=V \cup \{\mathscr{F} \in \mathscr{M} \mid V\in \mathscr{F}\}$. We shall call X' the *completely Hausdorff-completion* of X. In general, if T is an extension space of a topological space S, the *trace filters* of T are the filters $\mathscr{T}(t)$, $t \in T-S$, where $\mathscr{T}(t)$ is the filter on S generated by the traces $U \cap S$ of the neighborhoods $U\subset T$ of t. In case S is completely Hausdorff and $T=S'$, $\mathscr{T}(\mathscr{F})=\mathscr{F}$ if $\mathscr{F} \in S'-S$, i.e., the trace filters of S' are the maximal completely regular filters $\mathscr{F}$ on S such that $\bigcap \{G \mid G \in \mathscr{F}\}=\varnothing$.

If X and Z are topological spaces, we shall denote by $C(X, Z)$ the set of all continuous mappings of X into Z.

Suppose that Y is a completely Hausdorff space which is completely regular. Then Y' is the Stone-Čech compactification of Y (see [1]). It is well known that Y' has the following properties: if Z is a compact Hausdorff space, then each function in $C(Y, Z)$ has a unique extension in $C(Y', Z)$; Y' is locally connected if and only if Y is locally connected and pseudocompact [11]; Y' is connected if and only if Y is connected; $C(Y)$ and $C(Y')$ are isomorphic, and if R is the set of all real numbers, $C(Y')$ and $C(Y, R)$ are isomorphic only if Y is pseudocompact. The next theorem shows that almost all of these properties hold for Y' if Y is completely Hausdorff, but not necessarily completely regular.

THEOREM 4. *Let X be a completely Hausdorff space. The completely Hausdorff-completion X' of X has the following properties.*

(i) *If Z is a compact Hausdorff space, then each function in $C(X, Z)$ has a unique extension in $C(X', Z)$.*

(ii) *The Stone-Weierstrass theorem holds for X'.*

(iii) *X' is locally connected if and only if X is locally connected and each trace filter has a base consisting of connected open sets.*

(iv) *X' is not locally connected unless X is locally connected and pseudocompact.*

(v) *X' is connected if and only if X is connected.*

(vi) *$C(X')$ and $C(X)$ are isomorphic, and if R is the set of all real numbers, $C(X')$ and $C(X, R)$ are isomorphic only if X is pseudocompact.*

(vii) *If (Y, h) is an extension space of X such that Y is completely Hausdorff-complete, each element of $C(h(X))$ has an extension in $C(Y)$, and each trace filter of Y is a completely regular filter on $h(X)$, then there is a one-to-one function $g \in C(Y, X')$ for which $g(Y) = X'$ and $g \circ h$ is the identity on X.*

Proof. (i) Let $f \in C(X, Z)$. By almost the same argument as one given in the proof of Theorem 1, one can show that $f(\mathscr{G})$ is a convergent filter base if $\mathscr{G}$ is a maximal completely regular filter on X. Define f' by $f'(x) = f(x)$ if $x \in X$, and $f'(\mathscr{H}) = \lim f(\mathscr{H})$ if $\mathscr{H} \in X' - X$. Take $\mathscr{F} \in X' - X$, and choose open sets O and P such that $f'(\mathscr{F}) \in O \subset \bar{O} \subset P$. As $f'(\mathscr{F}) = \lim f(\mathscr{F})$, there is a set $V \in \mathscr{F}$ open in X such that $f(V) \subset O$. Necessarily the open neighborhood V^* of $\mathscr{F}$ has the property that $f'(V^*) \subset P$, for suppose there is a filter $\mathscr{G} \in V^*$ such that $\lim f(\mathscr{G}) \notin \bar{O}$: there exists an open set $f'(\mathscr{G}) \in W$, with $O \cap W = \varnothing$; $f^{-1}(W) \in \mathscr{G}$ since $f(\mathscr{G})$ converges to $f'(\mathscr{G})$; also $V \in \mathscr{G}$ since $\mathscr{G} \in V^*$; but then $\varnothing = V \cap f^{-1}(W) \in \mathscr{G}$, which is impossible. Therefore, f' is continuous at $\mathscr{F}$. The proof that f' is continuous at an arbitrary point of X is similar. Thus $f' \in C(X', Z)$. Clearly $f'|X = f$. Since X is dense in X', and the space Z is Hausdorff, f' is unique.

(ii) Since X is completely Hausdorff, and since by (i) each function in $C(X)$ has an extension in $C(X')$, $C(X')$ separates the points of X. If $x \in X$, and if $\mathscr{F} \in X' - X$, then because $\bigcap \{F \mid F \in \mathscr{F}\} = \varnothing$, we can choose a function $f \in C(X)$ such that

$f(x)=1$, and $f(F)=0$, some $F\in\mathscr{F}$; the extension f' of f in $C(X')$ has the property that $f'(x)=1\neq 0=f'(\mathscr{F})$. $C(X')$ also separates the points of $X'-X$, for suppose that $\mathscr{G}$, $\mathscr{H}\in X'-X$, $\mathscr{G}\neq\mathscr{H}$: as $\mathscr{G}$ and $\mathscr{H}$ are distinct maximal completely regular filters on X, there exist sets $G\in\mathscr{G}$ and $H\in\mathscr{H}$ such that $G\cap H=\varnothing$; furthermore, G and H can be chosen so that there is a function $g\in C(X)$ such that $g(G)=0$, and $g(H)=1$; then the extension g' of g in $C(X')$ satisfies $g'(\mathscr{G})=0\neq 1=g'(\mathscr{H})$. Hence X' is completely Hausdorff.

Let $\mathscr{F}$ be a completely regular filter on X'. We wish to show that $\bigcap\{F \mid F\in\mathscr{F}\}\neq\varnothing$. If there is a point $x\in X$ such that $x\in\bigcap\{F\mid F\in\mathscr{F}\}$, we are done. Suppose that $\bigcap\{F\cap X\mid F\in\mathscr{F}\}=\varnothing$, and let $\mathscr{G}$ be the filter on X generated by $\{F\cap X\mid F\in\mathscr{F}\}$. Then $\mathscr{G}$ is completely regular, so there is a maximal completely regular filter $\mathscr{H}$ on X such that $\mathscr{G}\subset\mathscr{H}$, and $\bigcap\{H\mid H\in\mathscr{H}\}=\varnothing$. Then $\mathscr{H}\in X'-X$, and since $\mathscr{G}\subset\mathscr{H}$, $F\cap X\cap H\neq\varnothing$, for every $F\in\mathscr{F}$ and $H\in\mathscr{H}$. Therefore, $F\cap H^*\neq\varnothing$, for every $F\in\mathscr{F}$ and $H\in\mathscr{H}$. This implies $\mathscr{H}$ is in the adherence of $\mathscr{F}$, for $\{V^*\mid V \text{ is open in } X\}$ is closed under the taking of finite intersections and so actually is a base for the topology of X'. As $\mathscr{F}$ is completely regular, $\varnothing\neq\bigcap\{F\mid F\in\mathscr{F}\}$.

(iii) A filter $\mathscr{C}$ on a topological space E is called *open* provided that it has a base consisting of open sets. An open filter $\mathscr{C}$ is called *connected* provided that whenever $O\cup P\in\mathscr{C}$, O and P disjoint open sets, either $O\in\mathscr{C}$ or $P\in\mathscr{C}$. In [4] it is shown that a maximal completely regular filter on a space E is connected. The principal result of [2] is: Let F be an extension space of E each of whose trace filters is connected. Then F is locally connected if and only if E is locally connected and each trace filter has a base consisting of connected open sets.

(iv) By (iii) X' is not locally connected unless X is locally connected. Suppose that X is not pseudocompact, and let Y be the completely regular Hausdorff space which has the same points as X, and whose topology is determined by $C(X)$, or, equivalently, $C(X, R)$, where R is the set of all real numbers. Then $C(X, R)=C(Y, R)$, so Y is not pseudocompact.

In [10] Glicksberg proved that if Z is a completely regular space, then the following are equivalent:

(a) Z is pseudocompact.

(b) For every sequence $\{V_n\}$ of nonempty open sets with disjoint closures, $\{\overline{V}_n\}$ has a cluster point, that is, a point x such that for every m and neighborhood V of x there exists an $n\geqq m$ for which $V\cap\overline{V}_n\neq\varnothing$.

In [11] it was noted that (b) is equivalent to the condition:

(c) For every sequence $\{V_n\}$ of nonempty open sets with disjoint closures, $\bigcup\{\overline{V}_n\}$ is not closed if $\{V_n\}$ is infinite. Therefore, since the space Y is completely regular, but not pseudocompact, it fails to satisfy (c).

Altering slightly Banaschewski's proof in [2] (that the Stone-Čech compactification of a completely regular Hausdorff space Z is not locally connected unless Z satisfies (c)), we shall show that X' cannot be locally connected. A corollary to Banaschewski's method of proof is: If Z is a completely regular space which does not satisfy (c), then there is a sequence of nonempty open sets $O(i, k) \subset Z$ $(i=1, 2, \ldots; k=1, 2, \ldots)$ with the following properties: for all $i, j, k, t, i \neq j$ implies $O(i, k) \cap O(j, t) = \varnothing$; for all $i, j, k, j \geqq k$ implies $O(i, j) \subset O(i, k)$; the filter $\mathscr{C}$ generated by the sets $\bigcup \{O(s, i) \mid s \geqq i\}$, $i=1, 2, \ldots$ is a completely regular filter on Z with empty adherence.

As Y is completely regular and does not satisfy (c), we may choose a filter $\mathscr{C}$ on Y and sets $O(i, k) \subset Y$ as above. Since X and Y have the same points, and since the topology of Y is weaker than the topology of X, $\mathscr{C}$ is a completely regular filter on X. $\bigcap \{C \mid C \in \mathscr{C}\} = \varnothing$, so there is a trace filter $\mathscr{F}$ of X' such that $\mathscr{C} \subset \mathscr{F}$. By an argument identical to one given in [2], it can be shown that $\mathscr{F}$ does not have a base consisting of connected open subsets of X: if F is a connected subset of X such that $F \subset \bigcup \{O(s, i) \mid s \geqq i\}$, some i, then necessarily $F \subset O(s, i)$, some $s \geqq i$, so

$$F \cap [\bigcup \{O(t, s+1) \mid t \geqq s+1\}] = \varnothing,$$

from which it follows that $F \notin \mathscr{F}$. By (iii) X' cannot be locally connected.

(v) is a consequence of (i) and the fact that X' is an extension space of X.

The known proof (see [9]) that (vi) holds for a completely regular Hausdorff space X also shows that (vi) holds for a completely Hausdorff space X which is not necessarily completely regular.

(vii) If $y \in Y - h(X)$, we shall denote by $\mathscr{V}(y)$ and $\mathscr{W}(y)$ the following filters: $\mathscr{V}(y)$ is the filter on Y generated by $\{U \mid U$ is open in Y, $y \in U$, and for some $f \in C(Y)$, $f(y)=0$, $f(Y-U)=1$, and $f(Y) \subset [0, 1]\}$; $\mathscr{W}(y)$ is the filter on $h(X)$ generated by $\{U \cap h(X) \mid U \in \mathscr{V}(y)\}$. If V is open in $h(X)$, $\tilde{V}$ is defined to be $V \cup \{y \in Y - h(X) \mid V \in \mathscr{W}(y)\}$. We shall show that the following hold: (a) for each $y \in Y - h(X)$, $\mathscr{V}(y)$ is a maximal completely regular filter on Y; (b) for each $y \in Y - h(X)$, $\mathscr{W}(y)$ is a maximal completely regular filter on $h(X)$; (c) the function e defined by $e(y) = h^{-1}(\mathscr{W}(y))$ is a one-to-one mapping of $Y - h(X)$ onto the set of all trace filters of X'; (d) if V is open in $h(X)$, then $\tilde{V}$ is open in Y; (e) if V is open in X, then for each $y \in [h(V)]^{\sim} - h(V)$, $e(y) \in V^*$; (f) the function g defined by $g(h(x)) = x$ if $x \in X$, and $g(y) = e(y)$ if $y \in Y - h(X)$ is a one-to-one continuous mapping of Y onto X'.

(a) Let $y \in Y - h(X)$, and suppose that $\mathscr{F}$ is a completely regular filter on Y such that $\mathscr{V}(y) \subset \mathscr{F}$. Since Y is completely Hausdorff-complete, $\bigcap \{F \mid F \in \mathscr{F}\} \neq \varnothing$. Since Y is completely Hausdorff, $\{y\} = \bigcap \{V \mid V \in \mathscr{V}(y)\}$. As $\bigcap \{F \mid F \in \mathscr{F}\} \subset \bigcap \{V \mid V \in \mathscr{V}(y)\}$, $y \in \bigcap \{F \mid F \in \mathscr{F}\}$, so $\mathscr{F} \subset \mathscr{V}(y)$.

(b) Let $y \in Y - h(X)$, and suppose that $\mathscr{G}$ is a completely regular filter on $h(X)$ such that $\mathscr{W}(y) \subset \mathscr{G}$. Take $G \in \mathscr{G}$, and choose a set $G' \subset G$ in $\mathscr{G}$ and a function $f \in C(h(X))$ such that $f(h(X)) \subset [0, 1]$, $f(G') = 0$, and $f(h(X) - G) = 1$. Let $g \in C(Y)$, $g \mid h(X) = f$. Since $\mathscr{W}(y) \subset \mathscr{G}$, $G' \cap U \neq \varnothing$ for all $U \in \mathscr{V}(y)$, so if $0 < t$, $g^{-1}([0, t)) \cap U \neq \varnothing$ for all $U \in \mathscr{V}(y)$. As $\mathscr{V}(y)$ is a maximal completely regular filter on Y, $g^{-1}([0, t)) \in \mathscr{V}(y)$ for each $t > 0$. Therefore, $G \in \mathscr{W}(y)$, for $G \supset f^{-1}([0, \frac{1}{2})) = h(X) \cap g^{-1}([0, \frac{1}{2})) \in \mathscr{W}(y)$, so $\mathscr{G} \subset \mathscr{W}(y)$.

(c) Let $y \in Y-h(X)$. By (b) $\mathscr{W}(y)$ is a maximal completely regular filter on $h(X)$. Since Y is completely Hausdorff, $\bigcap \{W \mid W \in \mathscr{W}(y)\} = \varnothing$. As h^{-1} is a homeomorphism of $h(X)$ onto X, $h^{-1}(\mathscr{W}(y))$ is a maximal completely regular filter on X, and $\varnothing = \bigcap \{W \mid W \in h^{-1}(\mathscr{W}(y))\}$. Hence $e(y)$ is a trace filter of X'.

Take $y, z \in Y-h(X)$, $y \neq z$. Since Y is completely Hausdorff, we may choose a function $f \in C(Y)$ such that $f(y) \neq f(z)$. For each $t>0$, let $O(t)=\{s \in Y \mid f(y)-t < f(s) < f(y)+t\}$. Then $O(t) \in \mathscr{V}(y)$, each $t>0$. Take $u>0$ so that $z \notin O(u)$, and choose a number $0<v<u$ and a function $g \in C(Y)$ such that $g(Y) \subset [0, 1]$, $g(O(v)) = 0$, and $g(Y-O(u))=1$. Define j by $j(s)=1-g(s)$ if $s \in Y$. Then $j \in C(Y)$, $j(Y) \subset [0, 1]$, $j(z)=0$, $j(Y-(Y-\bar{O}(v)))=1$, and $z \in Y-\bar{O}(v)$, so $Y-\bar{O}(v) \in \mathscr{V}(z)$. $O(v) \cap h(X) \in \mathscr{W}(y)$, $h(X)-\bar{O}(v) \in \mathscr{W}(z)$, and $O(v) \cap h(X) \cap (h(X)-\bar{O}(v)) = \varnothing$, so $\mathscr{W}(y) \neq \mathscr{W}(z)$. As h^{-1} is one-to-one, $e(y)=h^{-1}(\mathscr{W}(y)) \neq h^{-1}(\mathscr{W}(z))=e(z)$.

Let $\mathscr{F}$ be a trace filter of X'. We wish to show that there is a point $y \in Y-h(X)$ for which $e(y)=\mathscr{F}$. Since h is a homeomorphism, $h(\mathscr{F})$ is a maximal completely regular filter on $h(X)$, and $\varnothing = \bigcap \{F \mid F \in h(\mathscr{F})\}$. Let $Z=\{f \in C(h(X)) \mid f(h(X)) \subset [0, 1]$, and for some $G, G' \in h(\mathscr{F})$, $G' \subset G$, $f(G')=0$, and $f(h(X)-G)=1\}$. For each $f \in Z$, denote by f' the extension of f in $C(Y)$, and let $Z'=\{f' \mid f \in Z\}$. For each $f' \in Z'$ and $t>0$, let $V(f', t)=f'^{-1}([0, t))$, and let $\mathscr{G}$ be the filter on Y generated by the set of all finite intersections of elements of $\{V(f', t) \mid f' \in Z'$ and $t>0\}$. By an argument similar to one given in the proof of Theorem 1, $\mathscr{G}$ can be shown to be a completely regular filter on Y. Let $\mathscr{H}$ be a maximal completely regular filter on Y such that $\mathscr{G} \subset \mathscr{H}$. Since Y is completely Hausdorff-complete, $\bigcap \{H \mid H \in \mathscr{H}\} \neq \varnothing$. If $G \in h(\mathscr{F})$, $G \supset f^{-1}([0, \frac{1}{2}))=h(X) \cap V(f', \frac{1}{2})$, some $f \in Z$, so $h(X) \cap \bigcap \{H \mid H \in \mathscr{H}\} \subset \bigcap \{G \mid G \in \mathscr{G}\} \cap h(X) \subset \bigcap \{G \mid G \in h(\mathscr{F})\} = \varnothing$. Thus there is a point $y \in (Y-h(X)) \cap \bigcap \{H \mid H \in \mathscr{H}\}$. The maximality of $\mathscr{H}$ implies that $\mathscr{H} = \mathscr{V}(y)$. Since $f^{-1}([0, \frac{1}{2}))=h(X) \cap V(f', \frac{1}{2})$ and $V(f', \frac{1}{2}) \in \mathscr{V}(y)$ if $f \in Z$, $f^{-1}([0, \frac{1}{2})) \in \mathscr{W}(y)$, each $f \in Z$. Furthermore, if $G \in h(\mathscr{F})$, $G \supset f^{-1}([0, \frac{1}{2}))$, some $f \in Z$, so $G \in \mathscr{W}(y)$. Hence $h(\mathscr{F}) \subset \mathscr{W}(y)$, and since $h(\mathscr{F})$ is a maximal completely regular filter on $h(X)$, it follows that $h(\mathscr{F}) = \mathscr{W}(y)$. The point y then has the property that $e(y)=h^{-1}(\mathscr{W}(y))=h^{-1}(h(\mathscr{F}))=\mathscr{F}$.

(d) If $y \in Y-h(X)$, $\mathscr{W}(y)$ and the trace filter $\mathscr{T}(y)$ of Y are identical: by definition $\mathscr{W}(y) \subset \mathscr{T}(y)$; by hypothesis $\mathscr{T}(y)$ is a completely regular filter on $h(X)$; (b) then implies that $\mathscr{W}(y) = \mathscr{T}(y)$.

In general, if S is a topological space, if T is an extension space of S, and if V is open in S, then $V \cup \{t \in T-S \mid V \in \mathscr{T}(t)\}$ is open in T [2].

(e) If V is open in X, then for each $y \in [h(V)]^{\sim} - h(V)$, $h(V) \in \mathscr{W}(y)$, so $V = h^{-1}(h(V)) \in h^{-1}(\mathscr{W}(y)) = e(y)$, and hence $e(y) \in V^*$.

(f) Since h^{-1} is a one-to-one mapping of $h(X)$ onto X, (c) implies that g is a one-to-one mapping of Y onto X'. Let $y \in Y-h(X)$, and suppose that W is open in X', with $g(y) \in W$. Then there is a set V open in X such that $g(y) \in V^* \subset W$. By (d) $[h(V)]^{\sim}$ is open in Y, and $y \in [h(V)]^{\sim}$, for $g(y) \in V^*$ implies $V \in g(y)$ so that $h(V) \in \mathscr{W}(y)$. As a consequence of (e) and the definition of g, $g([h(V)]^{\sim}) \subset V^* \subset W$.

Thus g is continuous at y. The proof that g is continuous at each point of $h(X)$ is similar.

We conclude the proof of Theorem 4 with the remark that proofs of (i) and (ii) different from those given here can be obtained which depend on the properties of the Stone-Čech compactification Y' of the completely regular Hausdorff space Y whose points are those of X and whose topology is determined by $C(X)$.

The author does not know if the converse of (iv) in Theorem 4 holds, but, as the following example shows, *if X is a locally connected, pseudocompact, completely Hausdorff space, and if Y is a completely Hausdorff-complete extension space of X such that each function in $C(X)$ has an extension in $C(Y)$, then Y is not necessarily locally connected.*

EXAMPLE 5. The example given here is a slight modification of an unpublished one due to L. B. Treybig of a countably compact space $(Y, \mathscr{T})$, where Y is "the long interval", and $\mathscr{T}$ is a topology which is stronger than the usual order topology put on Y.

Description of the space. Let Ω be the first ordinal with an uncountable number of predecessors, let Ω' be the set of all ordinals less than Ω, and for each $x \in \Omega'$, let $I(x)$ be $\{x\} \times$ an open interval in the real line. Set $X = \Omega' \cup \bigcup \{I(x) \mid x \in \Omega'\}$, and for $x, y \in X$, define $x < y$ if (1) $x, y \in \Omega'$, and $x < y$ in Ω', or (2) $x \in \Omega'$, $y \in I(s)$, and $x \leqq s$ in Ω', or (3) $x \in I(r)$, $y \in \Omega'$, and $r < y$ in Ω', or (4) $x \in I(r)$, $y \in I(s)$, and $r < s$ in Ω', or (5) $x, y \in I(s)$, and $x < y$ in $I(s)$. Let $\mathscr{O}$ be the order topology on X. Let $Y = X \cup \{\Omega\}$, define $x < \Omega$ if $x \in X$, and let $\mathscr{R}$ be the order topology on Y. If $x \in X$, $O(x)$ will denote $\{y \in Y \mid x < y\}$. $\mathscr{T}$ will denote the topology on Y which is generated by $\{B \mid B \in \mathscr{R}$, or for some $x \in X$, $B = O(x) - \Omega'\}$.

The spaces $(X, \mathscr{O})$ and $(Y, \mathscr{T})$ have the following properties.

(i) *$(X, \mathscr{O})$ is a countably compact (hence pseudocompact), locally connected, completely regular Hausdorff space.*

(ii) *$(Y, \mathscr{T})$ is an extension space of $(X, \mathscr{O})$.*

(iii) *The Stone-Weierstrass theorem holds for $(Y, \mathscr{T})$.*

(iv) *Every continuous real valued function on $(X, \mathscr{O})$ has an extension in $C((Y, \mathscr{T}))$.*

(v) *$(Y, \mathscr{T})$ is not locally connected.*

Proof. (i) is well known [13]. (ii) holds, for Ω' is closed in $(X, \mathscr{O})$, and $X \cap (O(x) - \Omega') \neq \varnothing$, each $x \in X$. As a consequence of the fact that each function in $C(\Omega')$ is eventually constant [9], $(Y, \mathscr{R})$ is the Stone-Čech compactification of $(X, \mathscr{O})$, so since $\mathscr{R} \subset \mathscr{T}$, and $(X, \mathscr{O})$ is pseudocompact, (iv) follows. (v) is obvious. We prove (iii).

As $(Y, \mathscr{R})$ is compact Hausdorff, and $\mathscr{R} \subset \mathscr{T}$, $(Y, \mathscr{T})$ is completely Hausdorff. To show that every completely regular filter on $(Y, \mathscr{T})$ has nonempty adherence, it suffices to show that every open filter on $(Y, \mathscr{T})$ has nonempty adherence.

Suppose that $\mathscr{F}$ is an open filter on $(Y, \mathscr{T})$ such that $\Omega \notin \bigcap \{\bar{F} \mid F \in \mathscr{F}\}$. Then there is a set $F \in \mathscr{F} \cap \mathscr{T}$ with the property that for some $x \in X$, $F \cap (O(x) - \Omega') = \varnothing$. Therefore, $F \cap [(O(x) - \Omega')]^- = \varnothing$. Clearly, each point of Ω' is a limit point of $O(x)-\Omega'$, so $\Omega'\subset[O(x)-\Omega']^-$. Hence $F\cap O(x)\subset F\cap[(O(x)-\Omega')]^-=\varnothing$. As $\varnothing\notin\mathscr{F}$, and $F\cap G\in\mathscr{F}$, each $G\in\mathscr{F}$, $G\cap(Y-O(x))\neq\varnothing$, each $G\in\mathscr{F}$. Since $Y-O(x)$ is compact, it follows that $\bigcap\{\bar{G}\cap(Y-O(x))\mid G\in\mathscr{F}\}\neq\varnothing$.

In [8] it is noted that a Hausdorff space is absolutely closed if and only if every open filter on it has nonempty adherence. As a consequence of the proof of (iii) and the fact that $(Y,\mathscr{T})$ is a one-point extension space of a countably compact space, one obtains the

COROLLARY 6. *An absolutely closed, countably compact, completely Hausdorff space is not necessarily minimal Hausdorff.*

As noted in [3], there does not exist a noncompact, completely regular Hausdorff space for which the Stone-Weierstrass theorem holds. It can be shown, however, that *there exists a noncompact regular space* (as used in this paper, the condition of regularity includes T_1 separation) *for which the Stone-Weierstrass theorem holds.*

An open filter is called *regular* if it has a base consisting of closed sets. A regular space is *regular closed* provided that it is closed in every regular space in which it can be embedded.

In [12] Herrlich proved that a regular space is regular closed if and only if each regular filter on it has nonempty adherence. He also showed that there is a regular space (a subspace of the minimal regular noncompact space constructed in [7]), which we shall denote by $(S,\mathscr{W})$, with the property that $(S,\mathscr{W})$ is regular closed, but not minimal regular. In particular, he showed that there exists a topology $\mathscr{V}\subset\mathscr{W}$, $\mathscr{V}\neq\mathscr{W}$, such that $(S,\mathscr{V})$ is a compact Hausdorff space.

As a consequence of the fact that every completely regular filter is a regular filter, every completely regular filter on $(S,\mathscr{W})$ has nonempty adherence. In addition, $(S,\mathscr{W})$ is completely Hausdorff, for $(S,\mathscr{V})$ is completely Hausdorff, and $\mathscr{V}\subset\mathscr{W}$. $(S,\mathscr{W})$ is thus *a noncompact regular closed space for which the Stone-Weierstrass theorem holds.*

Two questions which one might consider are the following:

(i) If the Stone-Weierstrass theorem holds for a regular space R, is R necessarily regular closed?

(ii) Does there exist a regular space R such that the Stone-Weierstrass theorem holds for R, but R is not second category?

As Example 8 will show, not every space for which the Stone-Weierstrass theorem holds is second category. If, however, the answer to (i) is yes, then as a consequence of Theorem 7, the answer to (ii) must be no.

THEOREM 7. *Every regular closed space is second category.*

Proof. A *regular filter base* is a filter base consisting of open sets which is equivalent to a filter base consisting of closed sets. In [7] it is shown that on a minimal regular space (α) every regular filter base which has a unique adherent point is convergent, and (β) every regular filter base has an adherent point. The proof given in [5] that a minimal regular space is second category depends only on the fact that (β) holds on a minimal regular space. Clearly, (β) holds on a topological space X if and only if every regular filter on X has an adherent point, and, as noted above, Herrlich has proved that every regular filter on a regular closed space has an adherent point.

EXAMPLE 8. Let $X=[0, 1]$, let $\mathscr{V}$ be the usual topology on X, let Q be the set of all rational numbers in X, and define $\mathscr{W}$ to be the weakest topology on X such that $\mathscr{V} \subset \mathscr{W}$ and $Q \in \mathscr{W}$.

As noted in [6], $(X, \mathscr{W})$ is an absolutely closed space which is not minimal Hausdorff. In addition, $\mathscr{V} \subset \mathscr{W}$, so $(X, \mathscr{W})$ is completely Hausdorff, and the Stone-Weierstrass theorem holds for $(X, \mathscr{W})$.

For each $q \in Q$, the set $F(q)=\{q\} \cup (X-Q)$ is a closed, nowhere dense set in $(X, \mathscr{W})$, and $X=\bigcup \{F(q) \mid q \in Q\}$, so $(X, \mathscr{W})$ is not second category.

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